

Equations of general relativistic radiation hydrodynamics in Kerr space-time

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Accepted 200X December 15. Received 200X December 14; in original form 200X October 11

ABSTRACT

Equations of fully general relativistic radiation hydrodynamics in Kerr space-time are derived. While the interactions between matter and radiation are introduced in the comoving frame, the derivatives used when describing the global evolutions of both the matter and the radiation are given in the Boyer-Lindquist frame (BLF) which is a frame fixed to the coordinate describing the central black hole. Around a rotating black hole, both the matter and the radiation are influenced by the frame-dragging effects due to the black hole's rotation. As a fixed frame, we use the locally non-rotating reference frame (LNRF) which is one of the orthonormal frame. While the special relativistic effects such as beaming effects are introduced by the Lorentz transformation connecting the comoving frame and the LNRF, the general relativistic effects such as frame-dragging and gravitational redshift are introduced by the tetrads connecting the LNRF and the BLF.

Key words: accretion: accretion disks—black hole physics—hydrodynamics—radiative transfer—relativity.

1 INTRODUCTION

Accretion disks around black holes play one of the essential roles in the active universe. When the mass accretion rate is near or over the Eddington mass accretion rate, the radiation in the disk interact with matter. For the supercritical accretion flows considered in e.g. Syfert galaxies or the black hole binaries, the photons interact with matter. On the other hand, for the hypercritical accretion flows considered as one of the leading candidate for the central engine of gamma-ray bursts, the neutrinos interact with matter around the central black hole. In such situations, the dynamics and the energy balance in matter and radiation are influenced by each other. The radiative transfer in the curved space is investigated by many authors (e.g. Lindquist 1966; Anderson & Spiegel 1972; Schmid-Burgk 1978; Thorne 1981; Schinder, Bludman & Piran 1988; Turolla & Nobili 1988; Anile & Romano 1992; Cardall & Mezzacappa 2003; Park 2006). Lindquist (1966) gives a general treatment of the radiation transfer equation and the radiation hydrodynamic equations derived from the zeroth and the first moments of the radiation transfer equations by using a comoving Lagrangian frame of reference. For the higher moments, Anderson & Spiegel (1972) and Thorne (1981) gives a detailed treatments. Especially, Thorne (1981) derive general relativistic moment equations up to an arbitrary order by introducing projected symmetric trace-free (PSTF) tensors. This formalism has been used in spherically symmetric problems (e.g. Flammang 1982, 1984; Turolla & Nobili 1988). Other forms of the moment equations are presented by Schinder (1988) and Schinder & Bludman (1989) in a static, spherical space-time by using a Lagrangian comoving coordinate system. In these formalisms based on the comoving frames, the physical quantities in terms of matter and radiation and the directional derivatives are described in the comoving frame.

On the other hand, Mihalas (1980) introduce the alternative approach by using the radiation hydrodynamic equations in the Eulerian framework where the derivatives become much simpler form in this framework. In this approach, the physical quantities in terms of matter and radiation and the derivatives were introduced in the frame fixed to the coordinate of the central object, e.g. a black hole, while the interactions between matter and radiation were calculated in the comoving frame.

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That is, the local processes like the interaction between the matter and the radiations are evaluated in the comoving frame, while the derivatives which are used for the calculations of the global dynamics of both the matter and the radiation are derived in the frame fixed to the coordinate describing the central object. In principle, this formalism can be extended to arbitrary space-time. Actually, by using the energy-momentum tensor in a covariant form, Park (1993) derived the radiation hydrodynamic equations for spherically symmetric systems, and recently, Park (2006) give the explicit expressions for the basic equations of the general relativistic radiation hydrodynamics in Schwarzschild space-time. So far, the formalism in the space-time around a rotating black hole which is usually considered for the black holes in the real world have not been presented. One of the natural next step is to derive the explicit expressions for the general relativistic radiation hydrodynamics in space-time around a rotating black hole in this formalism. In this paper, we assume $c = 1$ in most equations except in a few cases where c is explicitly used for clarity. After giving the general forms of the basic equations for the general relativistic radiation hydrodynamics in §2, in §3 by using the orthonormal tetrads fixed to the coordinate (§3.1), the radiation moments (§3.2), the radiation four-force (§3.3) and the radiation hydrodynamic equations (§3.4) in Kerr space-time are derived. Concluding remarks are given in the last section.

2 BASIC EQUATIONS FOR GENERAL RELATIVISTIC RADIATION HYDRODYNAMIC IN COVARIANT FORM

Here, we summarize the basic equations for the general relativistic radiation hydrodynamic in covariant form (e.g. Mihalas & Mihalas 1984).

The energy-momentum tensor for matter of an ideal gas is described as

$$T^{\alpha\beta} = \rho_0 h_g u^\alpha u^\beta + P_g g^{\alpha\beta}, \quad (1)$$

where u^α , ρ_0 , h_g and P_g are the four-velocity, the rest-mass density, the relativistic specific enthalpy and the pressure of the gas, respectively. The specific enthalpy of the gas, h_g , is calculated as $h_g = (\varepsilon_g + P_g)/\rho_0$ where ε_g is the energy density of the gas. Here, the fluid quantities h_g , ε_g , P_g and ρ_0 are all being measured in the comoving frame of the fluid. The radiation stress-energy tensor is given as

$$R^{\alpha\beta} = \int \int I(\mathbf{n}, \nu) n^\alpha n^\beta d\nu d\Omega, \quad (2)$$

where $I(x^\alpha; \mathbf{n}, \nu)$ is the specific intensity of photons moving in direction \mathbf{n} with the frequency ν . The photon four-momentum $p^\alpha = (h\nu)(1, \mathbf{n})$ and $n^\alpha \equiv p^\alpha/h\nu$.

The continuity equation, i.e. particle number conservation equation, in the absence of particle creation and annihilation is given

$$(nu^\alpha)_{;\alpha} = 0, \quad (3)$$

where n is the particle number density measured in the comoving frame. Here, n is related to ρ_0 as $n = \rho_0/m_g$ where m_g is the mass of the gas particle. On the other hand, the conservation equation for the total energy-momentum of gas plus radiation is given as

$$(T^{\alpha\beta} + R^{\alpha\beta})_{;\beta} = 0. \quad (4)$$

The radiation four-force density acting on the matter is given as

$$G^\alpha = \frac{1}{c} \int \int [\chi_\nu I(\mathbf{n}, \nu) - \eta_\nu] n^\alpha d\nu d\Omega, \quad (5)$$

where χ_ν and η_ν are the opacity and the emissivity, respectively. The invariant emissivity and invariant opacity are η_ν/ν^2 and $\nu\chi_\nu$, respectively. The dynamical equations for the matter and the radiation field are described as

$$T^{\alpha\beta}_{;\beta} = G^\alpha, \quad (6)$$

and

$$R^{\alpha\beta}_{;\beta} = -G^\alpha, \quad (7)$$

respectively.

3 RADIATION HYDRODYNAMICS IN KERR SPACE-TIME

In order to see the correspondence with the work by Park (2006) deriving the equations for the radiation hydrodynamics in the Schwarzschild space-time, in the present study we use the Boyer-Lindquist coordinate for the description of the rotating

black hole's space-time. The background geometry around the rotating black hole written by the Boyer-Lindquist coordinate is described as

$$\begin{aligned} ds^2 &= g_{\alpha\beta} dx^\alpha dx^\beta, \\ &= -\alpha^2 dt^2 + \gamma_{ij} (dx^i + \beta^i dt)(dx^j + \beta^j dt), \end{aligned} \quad (8)$$

where $i, j = r, \theta, \phi$ and the nonzero components of the lapse function α , the shift vector β^i and the spatial matrix γ_{ij} are given in the geometric unit as

$$\alpha = \sqrt{\frac{\Sigma\Delta}{A}}, \quad \beta^\phi = -\omega, \quad \gamma_{rr} = \frac{\Sigma}{\Delta}, \quad \gamma_{\theta\theta} = \Sigma, \quad \gamma_{\phi\phi} = \frac{A \sin^2 \theta}{\Sigma}. \quad (9)$$

Here, we use the geometric mass $m = GM/c^2$, $\Sigma = r^2 + a^2 \cos^2 \theta$, $\Delta = r^2 - 2Mr + a^2$ and $A = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta = \Sigma\Delta + 2mr(r^2 + a^2)$, where M is the black hole mass, G is the gravitational constant and c is the speed of light. The position of the outer and inner horizon, r_\pm , is calculated from $\Delta = 0$ as $r_\pm = m \pm (m^2 - a^2)^{1/2}$. The angular velocity of the frame dragging due to the black hole's rotation is calculated as $\omega = -g_{t\phi}/g_{\phi\phi} = 2mar/A$. When $a/m = 0$, we obtain $\alpha = \Gamma$, $\beta^\phi = 0$, $\gamma_{rr} = 1/\Gamma^2$, $\gamma_{\theta\theta} = r^2$ and $\gamma_{\phi\phi} = r^2 \sin^2 \theta$ where $\Gamma \equiv (1 - 2m/r)^{1/2}$ (Park 2006).

3.1 Orthonormal Tetrads

In the present study, we use three frames: (1) the Boyer-Lindquist frame (BLF) which is the frame based on the Boyer-Lindquist coordinate describing the metric and have been frequently used for the description of the global dynamics (e.g. Gammie & Popham 1998), (2) the locally non-rotating reference frame (LNRF) which is the orthonormal frame fixed to the coordinate and is called as a fixed frame in Park (2006), and (3) the comoving frame where the interactions between the matter and the radiation are introduced. As a fixed tetrad which is the orthonormal tetrad fixed with respect to the coordinates, we use the tetrad for the LNRF (Bardeen, Press & Teukolsky 1972; Frolov & Novikov 1998). This frame is also called as a zero angular momentum observer (ZAMO) frame. This frame corresponds the generalization of the fixed frame used in Park (2006) for the space-time around a rotating black hole. When using the Boyer-Lindquist coordinate, the LNRF is the frame where the fiducial observer in this frame is rotating around the black hole with the angular velocity ω of the frame-dragging due to the black hole's rotation with zero radial velocity. The base $\mathbf{e}_{\hat{\alpha}} = \partial/\partial x^{\hat{\alpha}}$ can be expressed by the coordinate base $\partial/\partial x^\mu$ as

$$\frac{\partial}{\partial \hat{t}} = \frac{1}{\alpha} \left(\frac{\partial}{\partial t} - \beta^\phi \frac{\partial}{\partial \phi} \right), \quad \frac{\partial}{\partial \hat{r}} = \frac{1}{\sqrt{\gamma_{rr}}} \frac{\partial}{\partial r}, \quad \frac{\partial}{\partial \hat{\theta}} = \frac{1}{\sqrt{\gamma_{\theta\theta}}} \frac{\partial}{\partial \theta}, \quad \frac{\partial}{\partial \hat{\phi}} = \frac{1}{\sqrt{\gamma_{\phi\phi}}} \frac{\partial}{\partial \phi}. \quad (10)$$

Here, $(x^{\hat{0}}, x^{\hat{1}}, x^{\hat{2}}, x^{\hat{3}}) = (\hat{t}, \hat{r}, \hat{\theta}, \hat{\phi})$.

In the LNRF, a fiducial observer who is fixed with respect to the coordinates see the fluid with the three velocity whose components are calculated as

$$\hat{v}^i = \frac{u^{\hat{i}}}{u^{\hat{t}}}, \quad (i = r, \theta, \phi) \quad (11)$$

where $u^{\hat{\alpha}}$ is the four-velocity in the LNRF. The hat denotes the physical quantities measured in the LNRF. The components of the three velocity are explicitly calculated as

$$\hat{v}^r = \frac{\sqrt{\gamma_{rr}}}{\alpha u^{\hat{t}}} u^r, \quad \hat{v}^\theta = \frac{\sqrt{\gamma_{\theta\theta}}}{\alpha u^{\hat{t}}} u^\theta, \quad \hat{v}^\phi = \frac{\sqrt{\gamma_{\phi\phi}}}{\alpha} (\Omega + \beta^\phi), \quad (12)$$

where $\Omega \equiv u^\phi/u^t$ is the angular velocity and we have used $u^{\hat{t}} = -u_{\hat{t}} = \alpha u^t$. The Lorentz factor $\hat{\gamma}$ for this three velocity is calculated as

$$\hat{\gamma} \equiv (1 - \hat{v}^2)^{-1/2} = \alpha u^t, \quad (13)$$

where $\hat{v}^2 = \mathbf{\hat{v}} \cdot \mathbf{\hat{v}} = \hat{v}_r^2 + \hat{v}_\theta^2 + \hat{v}_\phi^2$.

A tetrad base for the comoving frame $\partial/\partial x^{\bar{\alpha}}$ is calculated by the Lorentz transformation as

$$\frac{\partial}{\partial x^{\bar{\alpha}}} = \Lambda^{\hat{\beta}}_{\bar{\alpha}}(\mathbf{\hat{v}}) \frac{\partial}{\partial x^{\hat{\beta}}}, \quad (14)$$

where the bar denotes the physical quantities measured in the comoving frame. The components of the Lorentz transformation $\Lambda^{\hat{\alpha}}_{\bar{\beta}}(\mathbf{\hat{v}})$ are given as $\Lambda^{\hat{t}}_{\bar{t}} = \hat{\gamma}$, $\Lambda^{\hat{i}}_{\bar{t}} = \hat{\gamma} \hat{v}^i$, $\Lambda^{\hat{t}}_{\bar{j}} = \hat{\gamma} \hat{v}_j$ and $\Lambda^{\hat{i}}_{\bar{j}} = \delta^i_j + \hat{v}^i \hat{v}_j \hat{\gamma}^2 / (1 + \hat{\gamma})$ ($i, j = r, \theta, \phi$). Here, $(x^{\bar{0}}, x^{\bar{1}}, x^{\bar{2}}, x^{\bar{3}}) = (\bar{t}, \bar{r}, \bar{\theta}, \bar{\phi})$. The components of the base of the comoving tetrad $\partial/\partial x^{\bar{\alpha}}$ can be expressed by the coordinate base $\partial/\partial x^\alpha$ as

$$\frac{\partial}{\partial \bar{t}} = \frac{\hat{\gamma}}{\alpha} \frac{\partial}{\partial t} + \frac{\hat{\gamma} \hat{v}_r}{\sqrt{\gamma_{rr}}} \frac{\partial}{\partial r} + \frac{\hat{\gamma} \hat{v}_\theta}{\sqrt{\gamma_{\theta\theta}}} \frac{\partial}{\partial \theta} + \hat{\gamma} \left(\frac{\hat{v}_\phi}{\sqrt{\gamma_{\phi\phi}}} - \frac{\beta^\phi}{\alpha} \right) \frac{\partial}{\partial \phi},$$

$$\begin{aligned}
\frac{\partial}{\partial \bar{r}} &= \frac{\hat{\gamma} \hat{v}_r}{\alpha} \frac{\partial}{\partial t} + \frac{1}{\sqrt{\gamma_{rr}}} \left[1 + \hat{v}_r^2 \left(\frac{\hat{\gamma}^2}{\hat{\gamma} + 1} \right) \right] \frac{\partial}{\partial r} + \frac{\hat{v}_r \hat{v}_\theta}{\sqrt{\gamma_{\theta\theta}}} \left(\frac{\hat{\gamma}^2}{\hat{\gamma} + 1} \right) \frac{\partial}{\partial \theta} + \left[\frac{\hat{v}_r \hat{v}_\phi}{\sqrt{\gamma_{\phi\phi}}} \left(\frac{\hat{\gamma}^2}{\hat{\gamma} + 1} \right) - \frac{\beta^\phi}{\alpha} \hat{\gamma} \hat{v}_r \right] \frac{\partial}{\partial \phi}, \\
\frac{\partial}{\partial \bar{\theta}} &= \frac{\hat{\gamma} \hat{v}_\theta}{\alpha} \frac{\partial}{\partial t} + \frac{\hat{v}_r \hat{v}_\theta}{\sqrt{\gamma_{rr}}} \left(\frac{\hat{\gamma}^2}{\hat{\gamma} + 1} \right) \frac{\partial}{\partial r} + \frac{1}{\sqrt{\gamma_{\theta\theta}}} \left[1 + \hat{v}_\theta^2 \left(\frac{\hat{\gamma}^2}{\hat{\gamma} + 1} \right) \right] \frac{\partial}{\partial \theta} + \left[\frac{\hat{v}_\theta \hat{v}_\phi}{\sqrt{\gamma_{\phi\phi}}} \left(\frac{\hat{\gamma}^2}{\hat{\gamma} + 1} \right) - \frac{\beta^\phi}{\alpha} \hat{\gamma} \hat{v}_\theta \right] \frac{\partial}{\partial \phi}, \\
\frac{\partial}{\partial \bar{\phi}} &= \frac{\hat{\gamma} \hat{v}_\phi}{\alpha} \frac{\partial}{\partial t} + \frac{\hat{v}_r \hat{v}_\phi}{\sqrt{\gamma_{rr}}} \left(\frac{\hat{\gamma}^2}{\hat{\gamma} + 1} \right) \frac{\partial}{\partial r} + \frac{\hat{v}_\theta \hat{v}_\phi}{\sqrt{\gamma_{\theta\theta}}} \left(\frac{\hat{\gamma}^2}{\hat{\gamma} + 1} \right) \frac{\partial}{\partial \theta} + \left\{ \frac{1}{\sqrt{\gamma_{\phi\phi}}} \left[1 + \hat{v}_\phi^2 \left(\frac{\hat{\gamma}^2}{\hat{\gamma} + 1} \right) \right] - \frac{\beta^\phi}{\alpha} \hat{\gamma} \hat{v}_\phi \right\} \frac{\partial}{\partial \phi}.
\end{aligned} \tag{15}$$

In the same way, the inverse transformation from the base of the comoving tetrad to the coordinate base is calculated by using the inverse Lorentz transformation $\Lambda_{\hat{\beta}}^{\hat{\alpha}}(-\mathbf{v})$, and the coordinate base $\partial/\partial x^\alpha$ is calculated from the base of the comoving tetrad $\partial/\partial x^{\hat{\alpha}}$ as

$$\begin{aligned}
\frac{1}{\alpha} \frac{\partial}{\partial t} &= \left(\hat{\gamma} - \frac{\beta^\phi \sqrt{\gamma_{\phi\phi}}}{\alpha} \hat{\gamma} \hat{v}_\phi \right) \frac{\partial}{\partial \bar{t}} + \left[-\hat{\gamma} \hat{v}_r + \frac{\beta^\phi \sqrt{\gamma_{\phi\phi}}}{\alpha} \hat{v}_r \hat{v}_\phi \left(\frac{\hat{\gamma}^2}{\hat{\gamma} + 1} \right) \right] \frac{\partial}{\partial \bar{r}} + \left[-\hat{\gamma} \hat{v}_\theta + \frac{\beta^\phi \sqrt{\gamma_{\phi\phi}}}{\alpha} \hat{v}_\theta \hat{v}_\phi \left(\frac{\hat{\gamma}^2}{\hat{\gamma} + 1} \right) \right] \frac{\partial}{\partial \bar{\theta}} \\
&\quad + \left\{ -\hat{\gamma} \hat{v}_\phi + \frac{\beta^\phi \sqrt{\gamma_{\phi\phi}}}{\alpha} \left[1 + \hat{v}_\phi^2 \left(\frac{\hat{\gamma}^2}{\hat{\gamma} + 1} \right) \right] \right\} \frac{\partial}{\partial \bar{\phi}}, \\
\frac{1}{\sqrt{\gamma_{rr}}} \frac{\partial}{\partial r} &= -\hat{\gamma} \hat{v}_r \frac{\partial}{\partial \bar{t}} + \left[1 + \hat{v}_r^2 \left(\frac{\hat{\gamma}^2}{\hat{\gamma} + 1} \right) \right] \frac{\partial}{\partial \bar{r}} + \hat{v}_r \hat{v}_\theta \left(\frac{\hat{\gamma}^2}{\hat{\gamma} + 1} \right) \frac{\partial}{\partial \bar{\theta}} + \hat{v}_r \hat{v}_\phi \left(\frac{\hat{\gamma}^2}{\hat{\gamma} + 1} \right) \frac{\partial}{\partial \bar{\phi}}, \\
\frac{1}{\sqrt{\gamma_{\theta\theta}}} \frac{\partial}{\partial \theta} &= -\hat{\gamma} \hat{v}_\theta \frac{\partial}{\partial \bar{t}} + \hat{v}_r \hat{v}_\theta \left(\frac{\hat{\gamma}^2}{\hat{\gamma} + 1} \right) \frac{\partial}{\partial \bar{r}} + \left[1 + \hat{v}_\theta^2 \left(\frac{\hat{\gamma}^2}{\hat{\gamma} + 1} \right) \right] \frac{\partial}{\partial \bar{\theta}} + \hat{v}_\theta \hat{v}_\phi \left(\frac{\hat{\gamma}^2}{\hat{\gamma} + 1} \right) \frac{\partial}{\partial \bar{\phi}}, \\
\frac{1}{\sqrt{\gamma_{\phi\phi}}} \frac{\partial}{\partial \phi} &= -\hat{\gamma} \hat{v}_\phi \frac{\partial}{\partial \bar{t}} + \hat{v}_r \hat{v}_\phi \left(\frac{\hat{\gamma}^2}{\hat{\gamma} + 1} \right) \frac{\partial}{\partial \bar{r}} + \hat{v}_\theta \hat{v}_\phi \left(\frac{\hat{\gamma}^2}{\hat{\gamma} + 1} \right) \frac{\partial}{\partial \bar{\theta}} + \left[1 + \hat{v}_\phi^2 \left(\frac{\hat{\gamma}^2}{\hat{\gamma} + 1} \right) \right] \frac{\partial}{\partial \bar{\phi}}.
\end{aligned} \tag{16}$$

3.2 Radiation moments

The radiation energy density E , the radiation flux F^i and the radiation pressure tensor P^{ij} are defined as the zeroth, the first and the second moments of the specific intensity $I_\nu(x^\mu, \mathbf{n})$, respectively. We denote the radiation moments as

$$\hat{E} = \int \int \hat{I}_\nu d\nu d\hat{\Omega}, \quad \hat{F}^i = \int \int \hat{I}_\nu \hat{n}^i d\nu d\hat{\Omega}, \quad \hat{P}^{ij} = \int \int \hat{I}_\nu \hat{n}^i \hat{n}^j d\nu d\hat{\Omega}, \tag{17}$$

when measured in the LNRF, and

$$\bar{E} = \int \int \bar{I}_\nu d\nu d\bar{\Omega}, \quad \bar{F}^i = \int \int \bar{I}_\nu \bar{n}^i d\nu d\bar{\Omega}, \quad \bar{P}^{ij} = \int \int \bar{I}_\nu \bar{n}^i \bar{n}^j d\nu d\bar{\Omega}, \tag{18}$$

when measured in the comoving frame. Correspondingly, the radiation stress tensors for the LNRF and the comoving frame are given as

$$R^{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} \hat{E} & \hat{F}^r & \hat{F}^\theta & \hat{F}^\phi \\ \hat{F}^r & \hat{P}^{rr} & \hat{P}^{r\theta} & \hat{P}^{r\phi} \\ \hat{F}^\theta & \hat{P}^{r\theta} & \hat{P}^{\theta\theta} & \hat{P}^{\theta\phi} \\ \hat{F}^\phi & \hat{P}^{r\phi} & \hat{P}^{\theta\phi} & \hat{P}^{\phi\phi} \end{pmatrix}, \tag{19}$$

and

$$R^{\bar{\alpha}\bar{\beta}} = \begin{pmatrix} \bar{E} & \bar{F}^r & \bar{F}^\theta & \bar{F}^\phi \\ \bar{F}^r & \bar{P}^{rr} & \bar{P}^{r\theta} & \bar{P}^{r\phi} \\ \bar{F}^\theta & \bar{P}^{r\theta} & \bar{P}^{\theta\theta} & \bar{P}^{\theta\phi} \\ \bar{F}^\phi & \bar{P}^{r\phi} & \bar{P}^{\theta\phi} & \bar{P}^{\phi\phi} \end{pmatrix}, \tag{20}$$

respectively. The contravariant components of the radiation stress tensor $R^{\alpha\beta}$ are calculated from $R^{\hat{\alpha}\hat{\beta}}$ by the transformation as

$$R^{\alpha\beta} = \frac{\partial x^\alpha}{\partial x^{\hat{\mu}}} \frac{\partial x^\beta}{\partial x^{\hat{\nu}}} R^{\hat{\mu}\hat{\nu}}, \tag{21}$$

and explicitly given as

$$R^{\alpha\beta} = \begin{bmatrix} \frac{\hat{E}}{\alpha^2} & \frac{\hat{F}^r}{\alpha\sqrt{\gamma_{rr}}} & \frac{\hat{F}^\theta}{\alpha\sqrt{\gamma_{\theta\theta}}} & \frac{1}{\alpha} \left(\frac{\hat{F}^\phi}{\sqrt{\gamma_{\phi\phi}}} - \beta^\phi \frac{\hat{E}}{\alpha} \right) \\ \frac{\hat{F}^r}{\alpha\sqrt{\gamma_{rr}}} & \frac{\hat{P}^{rr}}{\gamma_{rr}} & \frac{\hat{P}^{r\theta}}{\sqrt{\gamma_{rr}\gamma_{\theta\theta}}} & \frac{1}{\sqrt{\gamma_{rr}}} \left(\frac{\hat{P}^{r\phi}}{\sqrt{\gamma_{\phi\phi}}} - \beta^\phi \frac{\hat{F}^r}{\alpha} \right) \\ \frac{\hat{F}^\theta}{\alpha\sqrt{\gamma_{\theta\theta}}} & \frac{\hat{P}^{r\theta}}{\sqrt{\gamma_{rr}\gamma_{\theta\theta}}} & \frac{\hat{P}^{\theta\theta}}{\gamma_{\theta\theta}} & \frac{1}{\sqrt{\gamma_{\theta\theta}}} \left(\frac{\hat{P}^{\theta\phi}}{\sqrt{\gamma_{\phi\phi}}} - \beta^\phi \frac{\hat{F}^\theta}{\alpha} \right) \\ \frac{1}{\alpha} \left(\frac{\hat{F}^\phi}{\sqrt{\gamma_{\phi\phi}}} - \beta^\phi \frac{\hat{E}}{\alpha} \right) & \frac{1}{\sqrt{\gamma_{rr}}} \left(\frac{\hat{P}^{r\phi}}{\sqrt{\gamma_{\phi\phi}}} - \beta^\phi \frac{\hat{F}^r}{\alpha} \right) & \frac{1}{\sqrt{\gamma_{\theta\theta}}} \left(\frac{\hat{P}^{\theta\phi}}{\sqrt{\gamma_{\phi\phi}}} - \beta^\phi \frac{\hat{F}^\theta}{\alpha} \right) & \frac{\hat{P}^{\phi\phi}}{\gamma_{\phi\phi}} - \frac{2\beta^\phi}{\alpha\sqrt{\gamma_{\phi\phi}}} \hat{F}^\phi + \left(\frac{\beta^\phi}{\alpha} \right)^2 \hat{E} \end{bmatrix}. \quad (22)$$

By using the Lorentz transformations, the radiation moments measured in the comoving frame are calculated from those measured in the LNRF as

$$R^{\bar{\alpha}\bar{\beta}} = \Lambda^{\bar{\alpha}}_{\bar{\mu}}(-v) \Lambda^{\bar{\beta}}_{\bar{\nu}}(-v) R^{\hat{\mu}\hat{\nu}}, \quad (23)$$

and explicitly given as (Mihalas & Mihalas 1984; Park 2006)

$$\bar{E} = \hat{\gamma}^2 (\hat{E} - 2\hat{v}_i \hat{F}^i + \hat{v}_i \hat{v}_j \hat{P}^{ij}), \quad (24)$$

$$\bar{F}^i = -\hat{\gamma}^2 \hat{v}^i \hat{E} + \hat{\gamma} \left[\delta_j^i + \left(\hat{\gamma} + \frac{\hat{\gamma}^2}{\hat{\gamma} + 1} \right) \hat{v}^i \hat{v}_j \right] \hat{F}^j - \hat{\gamma} \hat{v}_j \left(\delta_k^i + \frac{\hat{\gamma}^2}{\hat{\gamma} + 1} \hat{v}^i \hat{v}_k \right) \hat{P}^{jk}, \quad (25)$$

$$\bar{P}^{ij} = \hat{\gamma}^2 \hat{v}^i \hat{v}^j \hat{E} - \hat{\gamma} \left(\hat{v}^i \delta_k^j + \hat{v}^j \delta_k^i + 2 \frac{\hat{\gamma}^2}{\hat{\gamma} + 1} \hat{v}^i \hat{v}^j \hat{v}_k \right) \hat{F}^k + \left(\delta_k^i + \frac{\hat{\gamma}^2}{\hat{\gamma} + 1} \hat{v}^i \hat{v}_k \right) \left(\delta_l^j + \frac{\hat{\gamma}^2}{\hat{\gamma} + 1} \hat{v}^j \hat{v}_l \right) \hat{P}^{kl}. \quad (26)$$

Inversely, the radiation moments measured in the LNRF are calculated from those measured in the comoving frame as

$$R^{\hat{\alpha}\hat{\beta}} = \Lambda^{\hat{\alpha}}_{\hat{\mu}}(v) \Lambda^{\hat{\beta}}_{\hat{\nu}}(v) R^{\bar{\mu}\bar{\nu}}, \quad (27)$$

and explicitly given as

$$\hat{E} = \hat{\gamma}^2 (\bar{E} + 2\hat{v}_i \bar{F}^i + \hat{v}_i \hat{v}_j \bar{P}^{ij}), \quad (28)$$

$$\hat{F}^i = \hat{\gamma}^2 \hat{v}^i \bar{E} + \hat{\gamma} \left[\delta_j^i + \left(\hat{\gamma} + \frac{\hat{\gamma}^2}{\hat{\gamma} + 1} \right) \hat{v}^i \hat{v}_j \right] \bar{F}^j + \hat{\gamma} \hat{v}_j \left(\delta_k^i + \frac{\hat{\gamma}^2}{\hat{\gamma} + 1} \hat{v}^i \hat{v}_k \right) \bar{P}^{jk}, \quad (29)$$

$$\hat{P}^{ij} = \hat{\gamma}^2 \hat{v}^i \hat{v}^j \bar{E} + \hat{\gamma} \left(\hat{v}^i \delta_k^j + \hat{v}^j \delta_k^i + 2 \frac{\hat{\gamma}^2}{\hat{\gamma} + 1} \hat{v}^i \hat{v}^j \hat{v}_k \right) \bar{F}^k + \left(\delta_k^i + \frac{\hat{\gamma}^2}{\hat{\gamma} + 1} \hat{v}^i \hat{v}_k \right) \left(\delta_l^j + \frac{\hat{\gamma}^2}{\hat{\gamma} + 1} \hat{v}^j \hat{v}_l \right) \bar{P}^{kl}. \quad (30)$$

3.3 Radiation four-force density

The radiation four-force density measured in the comoving frame is given as (Mihalas & Mihalas 1984)

$$G^{\bar{\alpha}} = \frac{1}{c} \int \int (\bar{\chi} \bar{I}_{\bar{\nu}} - \bar{\eta}) n^{\bar{\alpha}} d\bar{\nu} d\bar{\Omega}, \quad (31)$$

where $\bar{\chi}$ and $\bar{\eta}$ are the mean opacity and the emissivity measured in the comoving frame, respectively. The time component $G^{\bar{t}}$ has the dimension c^{-1} times the net rate of the radiation energy per unit volume, and the spatial component $G^{\bar{i}}$ has the dimension of the net rate of the momentum exchange between the matter and the radiation. The time component of the radiation four-force density measured in the comoving frame can be calculated as (Mihalas & Mihalas 1984; Park 2006)

$$G^{\bar{t}} = \bar{\Gamma} - \bar{\Lambda} \quad (32)$$

where the heating function $\bar{\Gamma}$ and the cooling function $\bar{\Lambda}$ are defined as

$$\bar{\Gamma} \equiv \frac{1}{c} \int \int \bar{\chi} \bar{I}_{\bar{\nu}} d\bar{\nu} d\bar{\Omega}, \quad \bar{\Lambda} \equiv \frac{1}{c} \int \int \bar{\eta} d\bar{\nu} d\bar{\Omega}. \quad (33)$$

The components of the radiation force G^α is calculated from those in the comoving frame $G^{\bar{\alpha}}$ by the transformation

$$G^\alpha = \frac{\partial x^\alpha}{\partial x^{\bar{\beta}}} G^{\bar{\beta}}, \quad (34)$$

and explicitly given as

$$\begin{aligned}
G^t &= \frac{\hat{\gamma}}{\alpha} \left(G^{\bar{t}} + \hat{v}_i G^{\bar{i}} \right), \\
G^r &= \frac{1}{\sqrt{\gamma_{rr}}} \left(G^{\bar{r}} + \hat{\gamma} \hat{v}_r G^{\bar{t}} + \frac{\hat{\gamma}^2}{\hat{\gamma} + 1} \hat{v}_r \hat{v}_i G^{\bar{i}} \right), \\
G^\theta &= \frac{1}{\sqrt{\gamma_{\theta\theta}}} \left(G^{\bar{\theta}} + \hat{\gamma} \hat{v}_\theta G^{\bar{t}} + \frac{\hat{\gamma}^2}{\hat{\gamma} + 1} \hat{v}_\theta \hat{v}_i G^{\bar{i}} \right), \\
G^\phi &= \frac{G^{\bar{\phi}}}{\sqrt{\gamma_{\phi\phi}}} + \hat{\gamma} \left(\frac{\hat{v}_\phi}{\sqrt{\gamma_{\phi\phi}}} - \frac{\beta^\phi}{\alpha} \right) G^{\bar{t}} + \hat{\gamma} \left[\frac{\hat{\gamma} \hat{v}_\phi}{\sqrt{\gamma_{\phi\phi}(\hat{\gamma} + 1)}} - \frac{\beta^\phi}{\alpha} \right] \hat{v}_i G^{\bar{i}}.
\end{aligned} \tag{35}$$

3.4 Radiation hydrodynamic equations

3.4.1 Continuity equation

As a continuity equation, now we consider the particle number conservation, $(nu^\alpha)_{;\alpha} = 0$. This equation can be calculated as

$$\frac{\partial}{\partial t} \left(n\alpha u^t \sqrt{\gamma} \right) + \frac{\partial}{\partial x^i} \left(n\alpha u^i \sqrt{\gamma} \right) = 0, \tag{36}$$

where $\gamma \equiv \det \gamma_{ij}$. In the case of the Kerr metric written by the Boyer-Lindquist coordinate $\gamma = (\Sigma A / \Delta) \sin^2 \theta$.

3.4.2 Hydrodynamic equations

The relativistic Euler equations are obtained by the projection of the equation of the energy-momentum tensor $T^{\alpha\beta}_{;\beta} = G^\alpha$ on the specific directions by using the projection tensor $P^{\alpha\beta} = g^{\alpha\beta} + u^\alpha u^\beta$ as $P^\alpha_\beta T^{\beta\delta}_{;\delta} = P^\alpha_\beta G^\beta$. From this, we can obtain $\rho_0 h_g u^\alpha_{;\beta} u^\beta + g^{\alpha\beta} P_{g,\beta} + u^\alpha u^\beta P_{g,\beta} = G^\alpha + u^\alpha u_\beta G^\beta$.

The radial part of the Euler equation, i.e. momentum conservation for r -direction, is calculated as

$$\begin{aligned}
&\rho_0 h_g u^t \frac{\partial u^r}{\partial t} + \rho_0 h_g u^i \frac{\partial u^r}{\partial x^i} + \frac{\rho_0 h_g}{2} \left[(\ln \gamma_{rr})_{,r} (u^r)^2 + 2 (\ln \gamma_{rr})_{,\theta} u^r u^\theta - (\ln \gamma_{\theta\theta})_{,r} \frac{\gamma_{\theta\theta}}{\gamma_{rr}} (u^\theta)^2 \right] \\
&- \frac{\rho_0 h_g}{2} \frac{(u^t)^2}{\gamma_{rr}} \left[\gamma_{\phi\phi} (\ln \gamma_{\phi\phi})_{,r} (\Omega + \beta^\phi)^2 + 2\gamma_{\phi\phi} \beta_{,r}^\phi (\Omega + \beta^\phi) - 2\alpha^2 (\ln \alpha)_{,r} \right] + \frac{1}{\gamma_{rr}} \frac{\partial P_g}{\partial r} + u^r \left(u^t \frac{\partial P_g}{\partial t} + u^i \frac{\partial P_g}{\partial x^i} \right) \\
&= -\alpha^2 u^t u^r G^t + [1 + \gamma_{rr} (u^r)^2] G^r + \gamma_{\theta\theta} u^r u^\theta G^\theta + \gamma_{\phi\phi} (\Omega + \beta^\phi) (G^\phi + \beta^\phi G^t) u^t u^r,
\end{aligned} \tag{37}$$

where the derivatives for $\ln \alpha$, β^ϕ , $\ln \gamma_{rr}$, $\ln \gamma_{\theta\theta}$ and $\ln \gamma_{\phi\phi}$ with respect to r and θ are given in App. A. The terms including $\beta^\phi (= -\omega)$ explicitly show the effects of the frame-dragging due to the black hole's rotation. Here, we have used the relation $g_{tt,r} (u^t)^2 + 2g_{t\phi,r} u^t u^\phi + g_{\phi\phi,r} (u^\phi)^2 = (u^t)^2 [\gamma_{\phi\phi} (\ln \gamma_{\phi\phi})_{,r} (\Omega + \beta^\phi)^2 + 2\gamma_{\phi\phi} \beta_{,r}^\phi (\Omega + \beta^\phi) - 2\alpha^2 (\ln \alpha)_{,r}]$. This terms are related to the centrifugal force. Actually, this relation can be also calculated as $g_{tt,r} (u^t)^2 + 2g_{t\phi,r} u^t u^\phi + g_{\phi\phi,r} (u^\phi)^2 = (\ln \gamma_{\phi\phi})_{,r} \gamma_{\phi\phi} (u^t)^2 (\Omega - \Omega_-)(\Omega - \Omega_+)$ where Ω_\pm is calculated as

$$\begin{aligned}
\Omega_\pm &= -\frac{g_{t\phi,r}}{g_{\phi\phi,r}} \pm \sqrt{\left(\frac{g_{t\phi,r}}{g_{\phi\phi,r}} \right)^2 - \frac{g_{tt,r}}{g_{\phi\phi,r}}} \\
&= \frac{\pm \sqrt{m}/\sin \theta}{\sqrt{\frac{r\Sigma^2}{2r^2 - \Sigma} - m(r^2 + a^2 - \Sigma) + ma^2 \sin^2 \theta \pm \sqrt{ma} \sin^2 \theta}}.
\end{aligned} \tag{38}$$

These angular velocity Ω_\pm is the generalization of the Keplerian angular velocity $\Omega_K^\pm = \pm m^{1/2}/(r^{3/2} \pm am^{1/2})$, and in the limit of $\theta = \pi/2$, we obtain $\Omega_\pm = \Omega_K^\pm$. In the right hand side of Eq. (37), the terms not including the derivatives represents the gravitational acceleration and the centrifugal acceleration, respectively.

The Euler equation in θ -direction, i.e. momentum conservation in θ -direction is calculated as

$$\begin{aligned}
&\rho_0 h_g u^t \frac{\partial u^\theta}{\partial t} + \rho_0 h_g u^i \frac{\partial u^\theta}{\partial x^i} + \frac{\rho_0 h_g}{2} \left[(\ln \gamma_{\theta\theta})_{,\theta} (u^\theta)^2 + 2 (\ln \gamma_{\theta\theta})_{,r} u^r u^\theta - (\ln \gamma_{rr})_{,\theta} \frac{\gamma_{rr}}{\gamma_{\theta\theta}} (u^r)^2 \right] \\
&- \frac{\rho_0 h_g}{2} \frac{(u^t)^2}{\gamma_{\theta\theta}} \left[\gamma_{\phi\phi} (\ln \gamma_{\phi\phi})_{,\theta} (\Omega + \beta^\phi)^2 + 2\gamma_{\phi\phi} \beta_{,\theta}^\phi (\Omega + \beta^\phi) - 2\alpha^2 (\ln \alpha)_{,\theta} \right] + \frac{1}{\gamma_{\theta\theta}} \frac{\partial P_g}{\partial \theta} + u^\theta \left(u^t \frac{\partial P_g}{\partial t} + u^i \frac{\partial P_g}{\partial x^i} \right) \\
&= -\alpha^2 u^t u^\theta G^t + \gamma_{rr} u^r u^\theta G^r + [1 + \gamma_{\theta\theta} (u^\theta)^2] G^\theta + \gamma_{\phi\phi} (\Omega + \beta^\phi) (G^\phi + \beta^\phi G^t) u^t u^\theta.
\end{aligned} \tag{39}$$

In the same way, the Euler equation in ϕ -direction, i.e. momentum conservation in ϕ -direction is calculated as

$$\begin{aligned} & \rho_0 h_g u^t \frac{\partial u^\phi}{\partial t} + \rho_0 h_g u^i \frac{\partial u^\phi}{\partial x^i} + \rho_0 h_g u^t u^r \left\{ (\Omega + \beta^\phi) \left[(\ln \gamma_{\phi\phi})_{,r} + \frac{\beta_\phi}{\alpha^2} \beta_{,r}^\phi \right] - 2\beta^\phi (\ln \alpha)_{,r} + \beta_{,r}^\phi \right\} \\ & + \rho_0 h_g u^t u^\theta \left\{ (\Omega + \beta^\phi) \left[(\ln \gamma_{\phi\phi})_{,\theta} + \frac{\beta_\phi}{\alpha^2} \beta_{,\theta}^\phi \right] - 2\beta^\phi (\ln \alpha)_{,\theta} + \beta_{,\theta}^\phi \right\} + \frac{1}{\gamma_{\phi\phi}} \frac{\partial P_g}{\partial \phi} + \frac{\beta^\phi}{\alpha^2} \left(\frac{\partial P_g}{\partial t} - \beta^\phi \frac{\partial P_g}{\partial \phi} \right) \\ & + u^\phi \left(u^t \frac{\partial P_g}{\partial t} + u^i \frac{\partial P_g}{\partial x^i} \right) = -\alpha^2 u^t u^\phi G^t + \gamma_{rr} u^r u^\phi G^r + \gamma_{\theta\theta} u^\theta u^\phi G^\theta + G^\phi + \gamma_{\phi\phi} (\Omega + \beta^\phi) (G^\phi + \beta^\phi G^t) u^t u^\phi. \end{aligned} \quad (40)$$

The for the local energy conservation is obtained from $u_\alpha T^{\alpha\beta}_{;\beta} = u_\alpha G^\alpha$ calculated as

$$-nu^t \frac{\partial}{\partial t} \left(\frac{\rho_0 h_g}{n} \right) - nu^i \frac{\partial}{\partial x^i} \left(\frac{\rho_0 h_g}{n} \right) + u^t \frac{\partial P_g}{\partial t} + u^i \frac{\partial P_g}{\partial x^i} = -\alpha^2 u^t G^t + \gamma_{rr} u^r G^r + \gamma_{\theta\theta} u^\theta G^\theta + \gamma_{\phi\phi} (\Omega + \beta^\phi) (G^\phi + \beta^\phi G^t) u^t. \quad (41)$$

The right hand side of Eq. (41) is also calculated by the heating and cooling function defined in the comoving frame as

$$u_\alpha G^\alpha = -G^{\bar{t}} = \bar{\Lambda} - \bar{\Gamma}. \quad (42)$$

3.4.3 Radiation moment equations

The radiation moment equation $R^{\alpha\beta}_{;\beta} = -G^\alpha$ gives the equation for the energy density, the radiation flux and the radiation pressure tensor. The radiation energy equation is obtained from t -component of this equation $R^{t\alpha}_{;\alpha} = -G^t$ calculated as

$$\begin{aligned} & \frac{\partial R^{tt}}{\partial t} + \frac{1}{\alpha\sqrt{\gamma}} \left[\frac{\partial}{\partial r} \left(\alpha\sqrt{\gamma} R^{tr} \right) + \frac{\partial}{\partial \theta} \left(\alpha\sqrt{\gamma} R^{t\theta} \right) \right] + \frac{\partial R^{t\phi}}{\partial \phi} + (g^{tt} g_{tt,r} + g^{t\phi} g_{t\phi,r}) R^{tr} + (g^{tt} g_{t\phi,r} + g^{t\phi} g_{\phi\phi,r}) R^{r\phi} \\ & + (g^{tt} g_{tt,\theta} + g^{t\phi} g_{t\phi,\theta}) R^{t\theta} + (g^{tt} g_{t\phi,\theta} + g^{t\phi} g_{\phi\phi,\theta}) R^{\theta\phi} = -G^t. \end{aligned} \quad (43)$$

The radiation momentum equation in r -direction $R^{r\alpha}_{;\alpha} = -G^r$ is calculated as

$$\begin{aligned} & \frac{\partial R^{tr}}{\partial t} + \frac{1}{\alpha\sqrt{\gamma}} \left[\frac{\partial}{\partial r} \left(\alpha\sqrt{\gamma} R^{rr} \right) + \frac{\partial}{\partial \theta} \left(\alpha\sqrt{\gamma} R^{r\theta} \right) \right] + \frac{\partial R^{r\phi}}{\partial \phi} + \frac{1}{2} g^{rr} (g_{rr,r} R^{rr} + 2g_{rr,\theta} R^{r\theta} - g_{\theta\theta,r} R^{\theta\theta}) \\ & - \frac{1}{2} g^{rr} (g_{tt,r} R^{tt} + 2g_{t\phi,r} R^{t\phi} + g_{\phi\phi,r} R^{\phi\phi}) = -G^r. \end{aligned} \quad (44)$$

In the similar manner, the radiation momentum equation in θ -direction $R^{\theta\alpha}_{;\alpha} = -G^\theta$ is calculated as

$$\begin{aligned} & \frac{\partial R^{t\theta}}{\partial t} + \frac{1}{\alpha\sqrt{\gamma}} \left[\frac{\partial}{\partial r} \left(\alpha\sqrt{\gamma} R^{r\theta} \right) + \frac{\partial}{\partial \theta} \left(\alpha\sqrt{\gamma} R^{\theta\theta} \right) \right] + \frac{\partial R^{\theta\phi}}{\partial \phi} + \frac{1}{2} g^{\theta\theta} (g_{\theta\theta,\theta} R^{\theta\theta} + 2g_{\theta\theta,r} R^{r\theta} - g_{rr,\theta} R^{rr}) \\ & - \frac{1}{2} g^{\theta\theta} (g_{tt,\theta} R^{tt} + 2g_{t\phi,\theta} R^{t\phi} + g_{\phi\phi,\theta} R^{\phi\phi}) = -G^\theta. \end{aligned} \quad (45)$$

The radiation momentum equation in ϕ -direction $R^{\phi\alpha}_{;\alpha} = -G^\phi$ is calculated as

$$\begin{aligned} & \frac{\partial R^{t\phi}}{\partial t} + \frac{1}{\alpha\sqrt{\gamma}} \left[\frac{\partial}{\partial r} \left(\alpha\sqrt{\gamma} R^{r\phi} \right) + \frac{\partial}{\partial \theta} \left(\alpha\sqrt{\gamma} R^{\theta\phi} \right) \right] + \frac{\partial R^{\phi\phi}}{\partial \phi} + (g^{\phi\phi} g_{t\phi,r} + g^{t\phi} g_{tt,r}) R^{tr} + (g^{\phi\phi} g_{\phi\phi,r} + g^{t\phi} g_{t\phi,r}) R^{r\phi} \\ & + (g^{\phi\phi} g_{t\phi,\theta} + g^{t\phi} g_{tt,\theta}) R^{t\theta} + (g^{\phi\phi} g_{\phi\phi,\theta} + g^{t\phi} g_{t\phi,\theta}) R^{\theta\phi} = -G^\phi. \end{aligned} \quad (46)$$

Finally, by inserting the components of the radiation stress tensor given by Eq. (22) in to Eqs. (43), (44), (45) and (46), we obtain the radiation moment equations corresponding to Eqs. (49), (52), (53) and (54) in Park (2006) as

$$\begin{aligned} & \frac{\partial}{\partial t} \left(\frac{\hat{E}}{\alpha^2} \right) + \frac{1}{\alpha\sqrt{\gamma}} \frac{\partial}{\partial r} \left(\sqrt{\frac{\gamma}{\gamma_{rr}}} \hat{F}^r \right) + \frac{1}{\alpha\sqrt{\gamma}} \frac{\partial}{\partial \theta} \left(\sqrt{\frac{\gamma}{\gamma_{\theta\theta}}} \hat{F}^\theta \right) + \frac{\partial}{\partial \phi} \left[\frac{1}{\alpha} \left(\frac{\hat{F}^\phi}{\sqrt{\gamma_{\phi\phi}}} - \beta^\phi \frac{\hat{E}}{\alpha} \right) \right] \\ & + 2(\ln \alpha)_{,r} \frac{\hat{F}^r}{\alpha\sqrt{\gamma_{rr}}} + 2(\ln \alpha)_{,\theta} \frac{\hat{F}^\theta}{\alpha\sqrt{\gamma_{\theta\theta}}} - \frac{\beta_{,r}^\phi}{\alpha^2} \sqrt{\frac{\gamma_{\phi\phi}}{\gamma_{rr}}} \hat{F}^{r\phi} - \frac{\beta_{,\theta}^\phi}{\alpha^2} \sqrt{\frac{\gamma_{\phi\phi}}{\gamma_{\theta\theta}}} \hat{F}^{\theta\phi} = -G^t. \end{aligned} \quad (47)$$

$$\begin{aligned} & \frac{\partial}{\partial t} \left(\frac{\hat{F}^r}{\alpha\sqrt{\gamma_{rr}}} \right) + \frac{1}{\alpha\sqrt{\gamma}} \frac{\partial}{\partial r} \left(\alpha\sqrt{\gamma} \frac{\hat{P}^{rr}}{\gamma_{rr}} \right) + \frac{1}{\alpha\sqrt{\gamma}} \frac{\partial}{\partial \theta} \left(\alpha\sqrt{\gamma} \frac{\hat{P}^{r\theta}}{\sqrt{\gamma_{rr}\gamma_{\theta\theta}}} \right) + \frac{\partial}{\partial \phi} \left[\frac{1}{\sqrt{\gamma_{rr}}} \left(\frac{\hat{P}^{r\phi}}{\sqrt{\gamma_{\phi\phi}}} - \beta^\phi \frac{\hat{F}^r}{\alpha} \right) \right] \\ & + \frac{(\ln \alpha)_{,r}}{\gamma_{rr}} \hat{E} - \frac{\sqrt{\gamma_{\phi\phi}} \beta_{,r}^\phi}{\alpha\gamma_{rr}} \hat{F}^\phi + \frac{1}{2} (\ln \gamma_{rr})_{,r} \frac{\hat{P}^{rr}}{\gamma_{rr}} + (\ln \gamma_{rr})_{,\theta} \frac{\hat{P}^{r\theta}}{\sqrt{\gamma_{rr}\gamma_{\theta\theta}}} - \frac{(\ln \gamma_{\theta\theta})_{,r}}{2\gamma_{rr}} \hat{P}^{\theta\theta} - \frac{(\ln \gamma_{\phi\phi})_{,r}}{2\gamma_{rr}} \hat{P}^{\phi\phi} = -G^r. \end{aligned} \quad (48)$$

$$\begin{aligned} & \frac{\partial}{\partial t} \left(\frac{\hat{F}^\theta}{\alpha\sqrt{\gamma_{\theta\theta}}} \right) + \frac{1}{\alpha\sqrt{\gamma}} \frac{\partial}{\partial r} \left(\alpha\sqrt{\gamma} \frac{\hat{P}^{r\theta}}{\sqrt{\gamma_{rr}\gamma_{\theta\theta}}} \right) + \frac{1}{\alpha\sqrt{\gamma}} \frac{\partial}{\partial \theta} \left(\alpha\sqrt{\gamma} \frac{\hat{P}^{\theta\theta}}{\gamma_{\theta\theta}} \right) + \frac{\partial}{\partial \phi} \left[\frac{1}{\sqrt{\gamma_{\theta\theta}}} \left(\frac{\hat{P}^{\theta\phi}}{\sqrt{\gamma_{\phi\phi}}} - \beta^\phi \frac{\hat{F}^\theta}{\alpha} \right) \right] \end{aligned}$$

$$\begin{aligned}
& + \frac{(\ln \alpha)_{,\theta}}{\gamma_{\theta\theta}} \hat{E} - \frac{\sqrt{\gamma_{\phi\phi}\beta_{,\theta}^\phi}}{\alpha\gamma_{\theta\theta}} \hat{F}^\phi - \frac{1}{2} (\ln \gamma_{rr})_{,\theta} \frac{\hat{P}^{rr}}{\gamma_{\theta\theta}} + (\ln \gamma_{\theta\theta})_{,r} \frac{\hat{P}^{r\theta}}{\sqrt{\gamma_{rr}\gamma_{\theta\theta}}} + \frac{1}{2} (\ln \gamma_{\theta\theta})_{,\theta} \frac{\hat{P}^{\theta\theta}}{\gamma_{\theta\theta}} - \frac{1}{2} (\ln \gamma_{\phi\phi})_{,\theta} \frac{\hat{P}^{\phi\phi}}{\gamma_{\theta\theta}} = -G^\theta. \quad (49) \\
& \frac{\partial}{\partial t} \left[\frac{1}{\alpha} \left(\frac{\hat{F}^\phi}{\sqrt{\gamma_{\phi\phi}}} - \beta^\phi \frac{\hat{E}}{\alpha} \right) \right] + \frac{1}{\alpha\sqrt{\gamma}} \frac{\partial}{\partial r} \left[\frac{\alpha\sqrt{\gamma}}{\sqrt{\gamma_{rr}}} \left(\frac{\hat{P}^{r\phi}}{\sqrt{\gamma_{\phi\phi}}} - \beta^\phi \frac{\hat{F}^r}{\alpha} \right) \right] \\
& + \frac{1}{\alpha\sqrt{\gamma}} \frac{\partial}{\partial \theta} \left[\frac{\alpha\sqrt{\gamma}}{\sqrt{\gamma_{\theta\theta}}} \left(\frac{\hat{P}^{\theta\phi}}{\sqrt{\gamma_{\phi\phi}}} - \beta^\phi \frac{\hat{F}^\theta}{\alpha} \right) \right] + \frac{\partial}{\partial \phi} \left[\frac{\hat{P}^{\phi\phi}}{\gamma_{\phi\phi}} - \frac{2\beta^\phi}{\alpha\sqrt{\gamma_{\phi\phi}}} \hat{F}^\phi + \left(\frac{\beta^\phi}{\alpha} \right)^2 \hat{E} \right] \\
& + [\beta_{,r}^\phi - 2\beta^\phi (\ln \alpha)_{,r}] \frac{\hat{F}^r}{\alpha\sqrt{\gamma_{rr}}} + [\beta_{,\theta}^\phi - 2\beta^\phi (\ln \alpha)_{,\theta}] \frac{\hat{F}^\theta}{\alpha\sqrt{\gamma_{\theta\theta}}} + \left[(\ln \gamma_{\phi\phi})_{,r} + \frac{\beta_\phi \beta_{,r}^\phi}{\alpha^2} \right] \frac{\hat{P}^{r\phi}}{\sqrt{\gamma_{rr}\gamma_{\phi\phi}}} \\
& + \left[(\ln \gamma_{\phi\phi})_{,\theta} + \frac{\beta_\phi \beta_{,\theta}^\phi}{\alpha^2} \right] \frac{\hat{P}^{\theta\phi}}{\sqrt{\gamma_{\theta\theta}\gamma_{\phi\phi}}} = -G^\phi, \quad (50)
\end{aligned}$$

where the derivatives for $\ln \alpha$, β^ϕ , $\ln \gamma_{rr}$, $\ln \gamma_{\theta\theta}$ and $\ln \gamma_{\phi\phi}$ with respect to r and θ are given in App. A.

4 CONCLUDING REMARKS

As is well known, the moment equations which are truncated at the finite order do not constitute a complete system of equations. That is, the number of equations are smaller than the number of variables to be solved. Thus, we need the additional equations to close the system of the equations (Chandrasekhar 1960; Mihalas 1970; Pomraning 1973; Rybicki & Lightman 1979; Mihalas & Mihalas 1984; Shu 1991) such as the Eddington factors. In the diffusion limit where the photon mean-free path is much smaller than the characteristic length of the system, i.e. in the optically thick region, the so-called Eddington approximation $P^{ij} = (\delta^{ij}/3)E$ is valid. On the other hand, in the optically thin region, the radiation becomes anisotropic in general. So, in such case, the full angle-dependent radiative transfer equations should be solved, and sometimes the variable Eddington factors are introduced. Some authors have expressed the variable Eddington factors as a function of the optical depth. Additionally, it is pointed out that the influence of the flow velocity should be also taken into account when constructing the closure relations. These problems and their calculation methods are extensively studied in the past studies (Auer & Mihalas 1970; Hummer & Rybicki 1971; Tamazawa et al. 1975; Mihalas 1980; Mihalas & Mihalas 1984; Schinder & Bludman 1989; Nobili, Turolla & Zampieri 1993; Yin & Miller 1995; Kato, Fukue & Mineshige 1998; Fukue 2006). The similar analysis for the closure relations will be studied in the case of the Kerr space-time in future. In the present study, we use the Boyer-Lindquist coordinate which is a frequently used coordinate in the astrophysics around a rotating black hole. However, there is a coordinate singularity at the horizon in this coordinate. Future studies also resolve this singularity by using the another coordinate with no singularity at the horizon (see, e.g. Takahashi 2007).

In this study, we have derived the radiation hydrodynamic equation in the Kerr space-time. While the interactions between the matter and the radiation are defined and calculated in the comoving frame, the derivatives which are used to describe the global evolutions of both the matter and the radiation are calculated in the BLF. Both the matter and the radiation are influenced by the frame-dragging due to the black hole's rotation. In this approach, as a fixed orthonormal frame, we use the so-called locally non-rotating reference frame (LNRF) which is firstly calculated in the Kerr space-time by Bardeen, Press & Teukolsky (1972) by using the Boyer-Lindquist coordinate. The special relativistic effects such as beaming effects are introduced when the Lorentz transformation between the LNRF and the comoving frame. On the other hand, the general relativistic effects such as frame-dragging effects due the black hole's rotation and the gravitational redshift are introduced by the tetrads describing the BLF.

ACKNOWLEDGMENTS

The author is grateful to Professors Y. Eriguchi and S. Mineshige for their continuous encouragements, and J. Fukue, Y. Sekiguchi, M. Shibata for useful discussion and comments, and J. Thiel for proofreading. The author also thanks to Department of Physics at Montana State University for its hospitality and Sachiko Tsuruta for her hospitality. This research was partially supported by the Ministry of Education, Culture, Sports, Science and Technology, Grant-in-Aid for Japan Society for the Promotion of Science (JSPS) Fellows (17010519).

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APPENDIX A: METRIC COMPONENTS AND DIFFERENTIAL VALUES

Here, we present the explicit expressions for the metric components and the their differential values with respect to r and θ used in this paper. Non-zero components of the metric are given as

$$\begin{aligned} g_{tt} &= -\alpha^2 + \beta_\phi \beta^\phi = -\left(1 - \frac{2mr}{\Sigma}\right), & g_{t\phi} &= \beta_\phi = -\frac{2mar \sin^2 \theta}{\Sigma}, & g_{\phi\phi} &= \gamma_{\phi\phi} = \frac{A \sin^2 \theta}{\Sigma}, \\ g_{rr} &= \gamma_{rr} = \frac{\Sigma}{\Delta}, & g_{\theta\theta} &= \gamma_{\theta\theta} = \Sigma, \end{aligned} \quad (A1)$$

and

$$\begin{aligned} g^{tt} &= -\frac{1}{\alpha^2} = -\frac{A}{\Sigma\Delta}, & g^{t\phi} &= \frac{\beta^\phi}{\alpha^2} = -\frac{2mar}{\Sigma\Delta}, & g^{\phi\phi} &= \gamma^{\phi\phi} - \frac{(\beta^\phi)^2}{\alpha^2} = \frac{1}{\Delta \sin^2 \theta} \left(1 - \frac{2mr}{\Sigma}\right), \\ g^{rr} &= \gamma^{rr} = \frac{\Delta}{\Sigma}, & g^{\theta\theta} &= \gamma^{\theta\theta} = \frac{1}{\Sigma}. \end{aligned} \quad (A2)$$

The differential values of the metric used in this paper are given as

$$\begin{aligned} (\ln \alpha)_{,r} &= \frac{1}{2} [(\ln \Sigma)_{,r} + (\ln \Delta)_{,r} - (\ln A)_{,r}], & \beta_{,r}^\phi &= -\omega \left[\frac{1}{r} - (\ln A)_{,r} \right], \\ (\ln \gamma_{rr})_{,r} &= (\ln \Sigma)_{,r} + (\ln \Delta)_{,r}, & (\ln \gamma_{\theta\theta})_{,r} &= (\ln \Sigma)_{,r}, & (\ln \gamma_{\phi\phi})_{,r} &= (\ln A)_{,r} - (\ln \Sigma)_{,r}, \\ (\ln \alpha)_{,\theta} &= \frac{1}{2} [(\ln \Sigma)_{,\theta} + (\ln \Delta)_{,\theta} - (\ln A)_{,\theta}], & \beta_{,\theta}^\phi &= \omega (\ln A)_{,\theta}, \\ (\ln \gamma_{rr})_{,\theta} &= (\ln \Sigma)_{,\theta} + (\ln \Delta)_{,\theta}, & (\ln \gamma_{\theta\theta})_{,\theta} &= (\ln \Sigma)_{,\theta}, & (\ln \gamma_{\phi\phi})_{,\theta} &= (\ln A)_{,\theta} + \frac{2}{\tan \theta} - (\ln \Sigma)_{,\theta}, \end{aligned} \quad (A3)$$

where

$$(\ln \Sigma)_{,r} = \frac{2r}{\Sigma}, \quad (\ln \Delta)_{,r} = \frac{2(r-m)}{\Delta}, \quad (\ln A)_{,r} = \frac{2}{A} [(r-m)\Sigma + (r+m)(r^2 + a^2)],$$

$$(\ln \Sigma)_{,\theta} = -a^2 \sin 2\theta, \quad (\ln \Delta)_{,\theta} = 0, \quad (\ln A)_{,\theta} = -\frac{a^2 \Delta}{A} \sin 2\theta. \quad (\text{A4})$$

Here, we also give the differential values of the metric as

$$\begin{aligned} g_{tt,r} &= \frac{2m}{\Sigma} \left(1 - \frac{2r^2}{\Sigma}\right), \quad g_{t\phi,r} = \frac{2ma \sin^2 \theta}{\Sigma} \left(\frac{2r^2}{\Sigma} - 1\right), \quad g_{\phi\phi,r} = 2 \sin^2 \theta \left[r - m + \frac{(3r^2 + a^2)m}{\Sigma} - \frac{2mr^2(r^2 + a^2)}{\Sigma^2}\right], \\ g_{rr,r} &= \frac{2}{\Delta} \left[r - \frac{(r-m)\Sigma}{\Delta}\right], \quad g_{\theta\theta,r} = 2r, \end{aligned} \quad (\text{A5})$$

and

$$\begin{aligned} g_{tt,\theta} &= \frac{2ma^2 r}{\Sigma^2} \sin 2\theta, \quad g_{t\phi,\theta} = -\frac{2mar(r^2 + a^2)}{\Sigma^2} \sin 2\theta, \quad g_{\phi\phi,\theta} = \left[\Delta + \frac{2mr(r^2 + a^2)^2}{\Sigma^2}\right] \sin 2\theta, \\ g_{rr,\theta} &= -\frac{a^2}{\Delta} \sin 2\theta, \quad g_{\theta\theta,\theta} = -a^2 \sin 2\theta. \end{aligned} \quad (\text{A6})$$

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